The von Kármán vortex street

A from-scratch incompressible Navier–Stokes solver, benchmark-verified and inverted into a flow-meter

Muhammet Emir Fil

B.Sc. Computational Engineering Science, RWTH Aachen • Technical report, June 21, 2026

Aim. Write a 2D incompressible Navier–Stokes solver from scratch, verify it against canonical benchmarks, and then invert it into a vortex flow-meter that reads the flow speed from the wake frequency. The goal is a forward PDE solver used backwards as a measurement, with the finite-domain error reported. The core is numpy-only.

Abstract

A 2D incompressible Navier–Stokes solver is written from scratch in numpy on a staggered (MAC) grid with Chorin projection, and used to model flow past a cylinder: the steady recirculation bubble, the von Kármán vortex street, and vortex-induced vibration (VIV) lock-in of an elastically mounted cylinder. The solver is verified against four classical benchmarks (Ghia et al. lid-driven cavity; Coutanceau–Bouard wake length; Williamson shedding Strouhal number; Khalak–Williamson VIV amplitude). It is then inverted: because a bluff body sheds at $f = \mathrm{St}\, U/D$, the wake frequency reads back the free-stream speed, which makes a vortex flow-meter, demonstrated as a simulate-then-recover round trip with a Monte-Carlo confidence interval. Finite-domain blockage shifts the force coefficients above their unbounded references; the offset is reported rather than tuned away, and cancels in the calibrated inverse.

1 Motivation

The same wake that makes a flag flutter and a chimney resonate is a clock: a bluff body in a steady stream sheds vortices at a frequency proportional to the flow speed. Building the solver that produces that wake from first principles, and then running it backwards to read the speed off the shedding frequency, exercises the full computational pipeline: discretising the incompressible Navier–Stokes equations, enforcing the divergence-free constraint, representing a solid body on a regular grid, validating against canonical benchmarks, and turning a verified forward model into a measurement. The core is numpy-only.

2 Method (forward solver)

Governing equations. The incompressible Navier–Stokes equations,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \nu\nabla^2\mathbf{u}, \qquad \nabla\cdot\mathbf{u} = 0, \tag{1}$$

are discretised on a staggered MAC grid (pressure at cell centres, velocities at face midpoints), which avoids the odd–even pressure decoupling of a collocated grid.

Projection. Time advance uses Chorin's projection method: an intermediate velocity $\mathbf{u}^\star$ is formed from advection and diffusion, then projected onto the divergence-free space by solving the pressure Poisson equation $\nabla^2 p = \frac{1}{\Delta t}\nabla\cdot\mathbf{u}^\star$. On the regular grid this is solved spectrally with a discrete cosine transform, which is fast and exact to round-off and keeps the iterative-solver cost out of the inner loop.

Immersed cylinder. The cylinder is represented by volume penalisation: a penalty term drives the velocity to zero inside the solid, and the integrated penalty reaction is the hydrodynamic force on the body (drag and lift), so the force coefficients come out of the same field that enforces the boundary. On a spring, the cylinder's transverse equation of motion is integrated alongside the flow, giving two-way fluid–structure coupling (VIV).

3 Verification against benchmarks

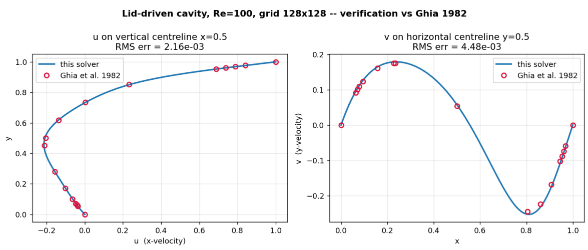
The solver is checked against four classical results spanning the regimes it claims (Table 1).

Table 1: Verification against canonical CFD benchmarks.

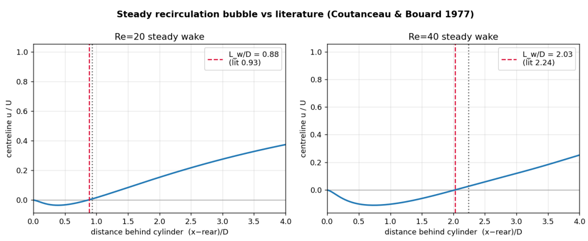
Check	This solver	Reference	
Lid-driven cavity centreline u, v (Re 100)	RMS $2.2/4.5 \times 10^{-3}$	Ghia, Ghia & Shin 1982	ok
Steady wake bubble L_w/D (Re 20 / 40)	0.88 / 2.03	0.93 / 2.24 (Coutanceau–Bouard)	ok
Shedding Strouhal St (Re 100)	0.18	0.16 unbounded (Williamson 1988)	+ blockage
Drag / lift C_d, C_l (Re 100)	1.56 / 0.42	$\sim$1.33 / $\sim$0.33 unbounded	+ blockage
VIV lock-in peak A/D	$\approx$ 0.5–0.6	Khalak & Williamson 1999	ok

For the lid-driven cavity, the centreline velocity profiles match the Ghia et al. (1982) tabulated data to RMS of order 10^{-3} (Fig. 1, left). The steady wake length and the VIV lock-in amplitude validate the immersed body and the fluid–structure coupling against the experimental literature.

Blockage. A finite domain confines the flow, raising St, C_d, and C_l above their unbounded references. This offset is quantified (St: 0.20 at 17 % blockage, 0.18 at 8 %, toward 0.16 as the walls recede) rather than fitted away.



(a) Lid-driven cavity centreline profiles vs Ghia et al. 1982.



(b) Steady recirculation bubble; length vs Coutanceau–Bouard.

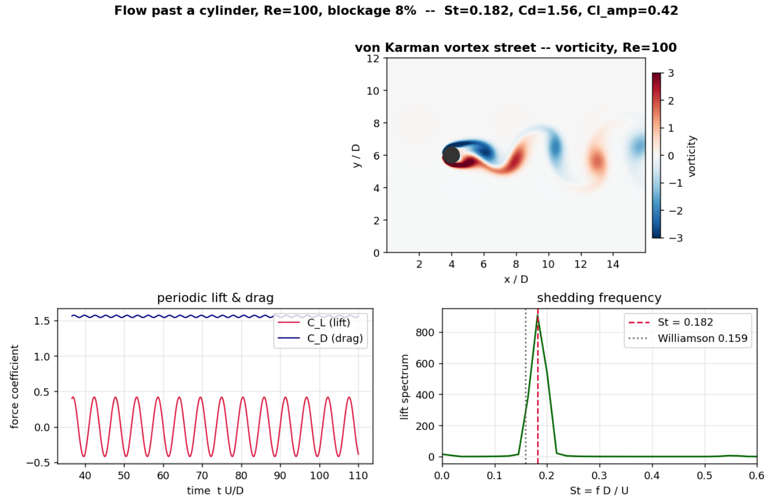
Figure 1: Verification: the cavity benchmark (left) and the steady wake (right).

4 The vortex street and VIV

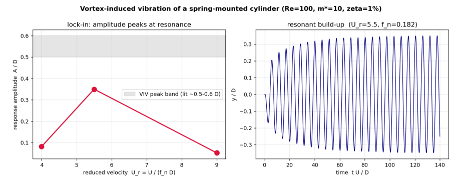
Above Re $\approx$ 47 the wake becomes unstable and sheds the periodic von Kármán vortex street (Fig. 2, left). When the cylinder is mounted on a spring, the alternating lift locks onto the structural frequency over a band of reduced velocities (vortex-induced vibration lock-in), and the amplitude rises to $A/D \approx 0.5$–0.6 at the peak (Fig. 2, right), consistent with the Khalak–Williamson experiments. This is the mechanism behind oscillating chimneys, risers, and heat-exchanger tubes.

5 Inversion: a vortex flow-meter

A bluff body sheds at $f = \mathrm{St}(\mathrm{Re})\, U/D$, so the wake frequency is a direct read-out of the flow speed, which is the principle of an industrial vortex flow-meter. The verified forward solver is



(a) Von Kármán vortex street, vorticity field at Re = 100.



(b) VIV lock-in amplitude vs reduced velocity.

Figure 2: The shedding wake (left) and the fluid–structure lock-in it drives (right).

inverted to demonstrate it: for a known free-stream U the solver sheds at f; reading f from the (noise-added) lift signal and inverting the solver-calibrated St(Re) relation recovers U, with a 95 % confidence interval from a sensor-noise Monte-Carlo. Because the same St(Re) is used forwards and backwards, the finite-domain blockage cancels in the round trip, exactly the way a real meter is calibrated per device.

6 Limitations

- 2D and laminar (Re up to a few hundred); a single circular cylinder.
- Volume penalisation gives $\mathcal{O}(\sqrt{\eta})$ boundary accuracy and a slightly thick effective cylinder.
- Finite-domain blockage shifts the force coefficients (quantified above; it cancels in the calibrated inverse).
- Explicit time stepping caps the step (CFL and diffusive stability).

7 Reproducibility

The core is numpy-only. Each result is reproduced by a script that prints its numbers and saves the figure shown here: `verify_cavity.py` (Ghia 1982), `verify_cylinder.py` (vortex street, St/C_d/C_l), `verify_cylinder_steady.py` (wake length), `verify_viv.py` (lock-in), and `verify_flowmeter.py` (the inverse recovery and CI). An interactive web demo animates a live vortex street with data injected (no server).

References

[1] U. Ghia, K. N. Ghia & C. T. Shin, "High-Re solutions for incompressible flow using the Navier–Stokes equations and a multigrid method," *J. Comput. Phys.*, 48, 1982.

[2] A. J. Chorin, "Numerical solution of the Navier–Stokes equations," *Math. Comp.*, 22, 1968. (The projection method.)

[3] C. H. K. Williamson, "Vortex dynamics in the cylinder wake," *Annu. Rev. Fluid Mech.*, 28, 1996; Strouhal–Reynolds relation (1988).

[4] M. Coutanceau & R. Bouard, steady wake recirculation length behind a circular cylinder, *J. Fluid Mech.*, 79, 1977.

[5] A. Khalak & C. H. K. Williamson, "Motions, forces and mode transitions in vortex-induced vibrations at low mass-damping," *J. Fluids Struct.*, 13, 1999.

[6] P. Angot, C.-H. Bruneau & P. Fabrie, "A penalization method to take into account obstacles in incompressible viscous flows," *Numer. Math.*, 81, 1999.