Time-step restrictions for explicit integration of damped moving-load beam problems, and two remedies

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Summary. When a damped Euler–Bernoulli beam under a moving load is marched with an explicit Runge–Kutta scheme, the stable time step is set neither by the load speed nor by the few physically relevant low modes: it is throttled by the *stiffness-proportional* part of Rayleigh damping acting on the highest, mesh-dependent finite-element modes. For a realistic 40 m highway overpass at 2 % damping this shrinks the explicit step by a factor of ≈ 13, which raises the cost of speed sweeps and Monte-Carlo studies. Two standard remedies remove the restriction at fixed accuracy: an implicit Newmark-β integrator (≈ $10^3\times$ larger step, identical dynamic amplification) and modal truncation (three modes reproduce the response to 0.16 % while enlarging the stable explicit step by ≈ 226×). Every result is checked against the Frýba closed-form solution and reproduced by a single numpy-only script.

1 Motivation

Vehicle–bridge interaction (VBI) studies routinely require not one simulation but *many*: the dynamic amplification factor (DAF) as a function of crossing speed, parameter sweeps over vehicle mass and damping, and Monte-Carlo ensembles for Bridge Weigh-in-Motion and drive-by monitoring. The per-run cost is therefore the practical bottleneck. Marching the semi-discrete equation of motion with an explicit integrator is attractive, being trivial to implement and needing no linear solve, but explicit schemes are only *conditionally* stable, and on a damped finite-element (FE) beam the condition is much more restrictive than for the undamped problem. This note isolates and quantifies that restriction on a concrete bridge and checks the two standard remedies against an analytical reference.

2 Model and integrators

A simply-supported uniform beam of span L, bending stiffness EI and mass per length $\bar{m}$ is discretised with two-node Euler–Bernoulli elements (transverse deflection and rotation per node, consistent mass), giving

$$\mathbf{M}\,\ddot{\mathbf{u}}(t) + \mathbf{C}\,\dot{\mathbf{u}}(t) + \mathbf{K}\,\mathbf{u}(t) = \mathbf{F}(t), \tag{1}$$

where $\mathbf{F}(t)$ is the consistent nodal load of a constant point force P moving at speed v, placed each step through the element shape functions. Damping is classical Rayleigh damping $\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}$, with α, β fixed by a target ratio ζ at the first two natural frequencies. Two beams are used throughout: a 20 m span for which the Frýba closed form is the exact ground truth, and a realistic 40 m overpass ($EI = 8.4 \times 10^{10}\,\mathrm{N\,m^2}$, $\bar{m} = 12\,000\,\mathrm{kg/m}$, first frequency $f_1 = 2.60\,\mathrm{Hz}$) for the stability and cost story.

Three integrators act on (1): *(i)* classic explicit RK4 on the first-order state-space form; *(ii)* implicit Newmark-β (average acceleration, $\gamma = \frac{1}{2}$, $\beta = \frac{1}{4}$); and *(iii)* RK4 on the modally reduced system (Section 4). All three share the same assembly and the same Frýba reference.

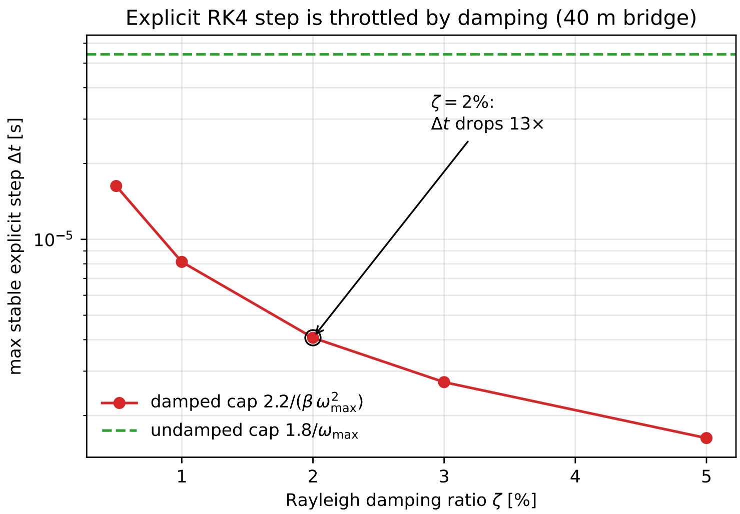

Figure 1: Maximum stable explicit (RK4) time step for the 40 m bridge as a function of the Rayleigh damping ratio. The undamped cap $1.8/\omega_{\max}$ (green) applies only at $\zeta = 0$; any stiffness-proportional damping moves the limit to the $2.2/(\beta\omega_{\max}^2)$ branch, which scales as $1/\zeta$ and is already 13× smaller at $\zeta = 2\,\%$.

3 The damping-dominated step restriction

RK4 is stable only where its growth factor satisfies $|R(\lambda\,\Delta t)| \le 1$ for every eigenvalue λ of the semi-discrete system. The relevant part of the stability region has two edges. For the *undamped* structure the eigenvalues are purely imaginary, $\lambda = \pm i\omega$, and RK4 is stable up to $\omega\,\Delta t \lesssim 2\sqrt{2} \approx 2.83$; the integrator therefore caps the step at $\Delta t \le 1.8/\omega_{\max}$ (a safety factor below 2.83), where $\omega_{\max}$ is the highest FE frequency. Because the consistent-mass beam has very stiff, mesh-dependent high modes, $\omega_{\max}$ is already large ($\approx 3.3 \times 10^4$ rad/s for the 40 m mesh).

Stiffness-proportional damping tightens the limit further. The modal form of (1) with $\mathbf{C} = \beta\mathbf{K}$ gives, for a mode of frequency ω,

$$\ddot{q} + \beta\omega^2\dot{q} + \omega^2 q = f(t), \qquad \lambda = \tfrac{1}{2}\left(-\beta\omega^2 \pm \sqrt{\beta^2\omega^4 - 4\omega^2}\right). \tag{2}$$

For the high modes $\omega \gg 2/\beta$ the discriminant is positive: the mode is *over-damped* and one root is a large negative real number, $\lambda \approx -\beta\omega^2$. RK4's stability region reaches only to ≈ -2.78 along the negative real axis, so these modes impose

$$\Delta t \;\le\; \frac{2.2}{\beta\,\omega_{\max}^2} \qquad \text{(real-axis limit, safety factor below 2.78).} \tag{3}$$

The ratio of the two caps is

$$\frac{\Delta t_{\text{undamped}}}{\Delta t_{\text{damped}}} = \frac{1.8/\omega_{\max}}{2.2/(\beta\omega_{\max}^2)} = 0.82\,\beta\,\omega_{\max}. \tag{4}$$

For the 40 m bridge at $\zeta = 2\,\%$ ($\beta = 4.90 \times 10^{-4}$ s, $\omega_{\max} = 3.32 \times 10^4$ rad/s) this is **13.3**: adding realistic damping cuts the explicit step from $\Delta t = 5.42 \times 10^{-5}$ s to 4.07×10^{-6} s (Figure 1). The step is now governed by $\beta\omega_{\max}^2$, a product of the damping model and the *mesh*, not of any physical time scale of the problem.

4 Two remedies

(a) Implicit integration (Newmark-β). The average-acceleration Newmark scheme is unconditionally stable for $\gamma \ge \frac{1}{2}$, $\beta \ge \frac{1}{4}$: the step is then chosen by *accuracy*, not stability. With a

Table 1: Damped 40 m bridge ($\zeta = 2\,\%$, $P = 300\,\mathrm{kN}$, $v = 25\,\mathrm{m/s}$): explicit RK4 versus implicit Newmark over one crossing. Same dynamic amplification, three orders of magnitude fewer steps.

integrator	Δt [s]	steps / crossing	DAF
explicit RK4	4.07×10^{-6}	393 039	1.0799
implicit Newmark	4.0×10^{-3}	400	1.0800
ratio	983×	983×	6.8×10^{-5} rel.

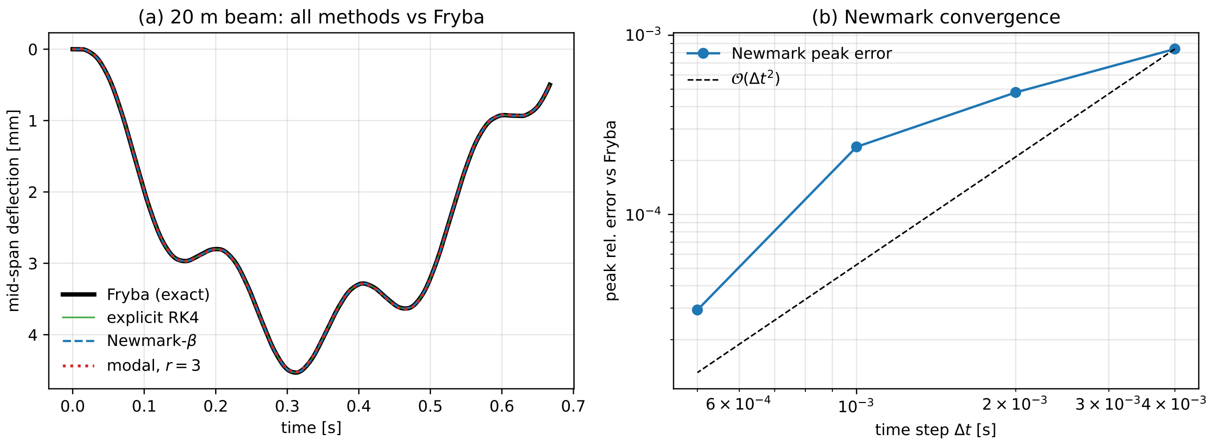

Figure 2: (a) Mid-span deflection of the 20 m beam at $v = 30\,\mathrm{m/s}$: explicit RK4, Newmark-β and the three-mode reduced model all sit on the Frýba exact curve. (b) Newmark peak-deflection error against Frýba versus step size, confirming the expected $\mathcal{O}(\Delta t^2)$.

fixed step its effective stiffness is factorised once. On the damped 40 m bridge Newmark runs at $\Delta t = 4 \times 10^{-3}$ s, **983**× larger than the explicit cap, and gives the same answer: DAF 1.0800 versus 1.0799, a relative difference of 6.8×10^{-5} (Table 1, Figure 3b). Accuracy against Frýba converges at the expected second order in Δt (Figure 2b).

(b) Modal truncation. Projecting (1) onto the first r mass-normalised eigenmodes, $\mathbf{u} = \mathbf{\Phi q}$ with $\mathbf{\Phi}^\top \mathbf{M} \mathbf{\Phi} = \mathbf{I}$ and $\mathbf{\Phi}^\top \mathbf{K} \mathbf{\Phi} = \mathrm{diag}(\omega_i^2)$, decouples the dynamics into r scalar oscillators. The reduced system's highest frequency is the retained ω_r, not $\omega_{\max}$, so by (3) the stable explicit step grows by $(\omega_{\max}/\omega_r)$ undamped and by $(\omega_{\max}/\omega_r)^2$ under stiffness-proportional damping. For the 20 m beam, $r = 3$ modes already reproduce the peak deflection to 0.16 % (Figure 3a) while reducing 40 free DOFs to 3 (13×) and enlarging the undamped stable step 226× ($\omega_{\max} = 7.27 \times 10^4\,\mathrm{rad/s} \to \omega_3 = 322\,\mathrm{rad/s}$). Modal truncation is the cheaper route when a handful of modes carry the response, as they do for a single-span bridge under traffic loads.

5 Verification

Every integrator is checked against the Frýba closed-form moving-force solution on the 20 m span (Figure 2a). At $v = 30\,\mathrm{m/s}$ the peak mid-span deflection matches Frýba to a relative error of 5.6×10^{-6} (explicit RK4), 2.4×10^{-4} (Newmark, $\Delta t = 1 \times 10^{-3}$ s) and 1.6×10^{-3} (modal, $r = 3$); the full-model DAF is 1.1417. These agreements establish that the large-step remedies of Section 4 change the *cost*, not the answer.

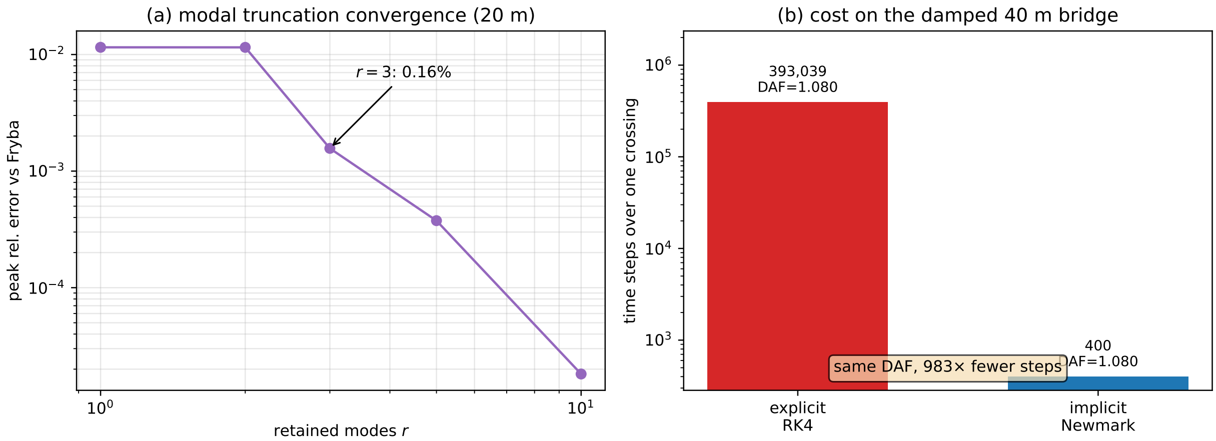

Figure 3: (a) Modal-truncation convergence on the 20 m beam: three modes reach 0.16 %. (b) Cost on the damped 40 m bridge: implicit Newmark reaches the same DAF as explicit RK4 with 983× fewer steps.

6 Limitations and scope

The model is deliberately the simplest one that exhibits the effect: a linear, single-span Euler–Bernoulli beam under a constant moving force with classical Rayleigh damping. The step restriction (3) is a known property of explicit integrators applied to stiffly damped systems; this note quantifies it in the moving-load setting and checks the two standard remedies against the Frýba reference. Mass-proportional damping ($\alpha\mathbf{M}$) does not cause the restriction; it is specific to the $\beta\mathbf{K}$ term. For nonlinear vehicle–bridge coupling or multi-span continuous girders the same reasoning applies to the linearised high-frequency content, but the constants in (4) would be re-derived. Both remedies are standard: Newmark-β is the textbook structural integrator and modal truncation is elementary model order reduction; either one avoids the step restriction.

Reproducibility

The beam FE core, the three integrators and the Frýba reference are numpy-only. Every figure and number in this note is regenerated by `python note/make_note_figures.py`, which writes `note/data.json`. Source: `github.com/bypire/vbi-bridge-sim`.

Acknowledgement of tools

AI assistance (Anthropic Claude) was used while preparing the code, figures and text of this note. Every reported number is produced by the accompanying `numpy`-only script; the author has checked the results and is responsible for their correctness.

References

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